

Instanton amplitudes in open-closed topological string theory

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ABSTRACT

I use the universal instanton formalism to discuss quantum effects in the open-closed topological string theory of a Calabi-Yau A-model, in the presence of a multiply-wrapped ‘Floer’ D-brane. This gives a precise meaning (up to the issue of compactifying the relevant moduli spaces) to the instanton corrections which affect sigma model and topological string amplitudes. The cohomological formalism I use recovers the homological approach used by Fukaya and collaborators in the singly-wrapped case, even though it is not a naive generalization of the latter. I also prove some non-renormalization theorems for amplitudes with low number of insertions. The non-renormalization argument is purely geometric and based on the universal instanton formulation, and thus it does not assume that the background satisfies the string equations of motion. These results are valid even though the D-brane background typically receives worldsheet instanton corrections. I also point out that the localized form of the boundary BRST operator receives instanton corrections and make a few comments on the consequences of this effect.

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1 Introduction

Topological sigma models in the presence of D-branes form an active area of research due to their relevance for open string extensions of mirror symmetry [13, 4, 20] and in general for the physics of D-branes in Calabi-Yau compactifications [11, 5]. Much recent work in this direction is based on the paper [8], where boundary conditions for such models were first considered. Unfortunately, recent studies of the subject suffer from a lack of systematic development of the basic framework of topological open-closed strings.

The purpose of the present paper is to carry out some of the required analysis for the open-closed topological A model and its associated string theory in the presence of a (multiply-wrapped) ‘Floer’ D-brane. This is a preliminary step which is necessary before one can consistently formulate the string field theory of open-closed topological A strings. A more detailed analysis of this theory is deferred to a companion paper [3].

Before discussing string theory issues, one has to consider the more basic problem of localization for topological sigma model and string amplitudes in the presence of disk instanton corrections. A clear formulation of localization does not seem to have been given for the multiply wrapped case¹. One of the aims of the present paper is to approach this problem from a ‘Lagrangian’ point of view (similar to the one adopted in [9, 10]), as opposed to the possibly more direct, but considerably less explicit Hamiltonian approach followed originally in [8]. This enables us to give a clear description of what the instanton amplitudes compute in the presence of a bundle on the D-brane’s worldvolume, and has the added advantage that it does not assume that the open string background satisfies the string equations of motion. As I discuss in more detail in [3], these results can be used to recover the correct description of on-shell observables in the presence of instanton corrections.

A precise formulation of instanton corrections to open string amplitudes was proposed in [6, 7]. Unfortunately, the homological approach of [6] seems to be limited to the singly-wrapped case, where Poincare duality can be used in its traditional sense. A naive extension of this approach is problematic for multiply wrapped D-branes, since Poincare duality cannot be used directly for bundle-valued forms. Instead, I will retreat to a cohomological formulation in terms of cup products of bundle valued forms

¹The definition of the ‘number of disk instantons’ is slightly non-obvious in this situation, and even the singly-wrapped case requires some careful analysis. Since the string field theory action of [8] does not take into account all of the relevant instanton effects, formal expressions obtained by differentiating the partition function of [8] as in [14, 12] can be misleading. This is true even for the case of D-branes wrapped once around *special* Lagrangian cycles, due to the existence of further instanton corrections which are responsible for the obstructions noticed in [6, 12, 7]. In the work of [13, 12], one is faced with the situation of recovering disk instanton amplitudes from a string field theory action which is in contradiction with the existence of these obstructions. Clearly one needs an independent analysis of instanton amplitudes and a construction of the string field theory which takes all such effects into account. This can be achieved by building the string field theory around a background which does not satisfy the equations of motion, as I shall discuss in more detail in [3].

defined over the instanton moduli space, which also has the advantage that it is more directly related to the physical interpretation of amplitudes. This requires a careful discussion of localization and the use of the universal instanton formalism, which is carried out for the open-closed sigma model in Section 3, and for the associated topological string theory in Section 4. The result reduces to that of [6, 7] for the particular case of singly-wrapped D-branes.

As noticed in [6] and rediscovered in [11, 12], the associated open string background will typically receive quantum corrections. This means that one is building the string field theory around a background which fails to satisfy the instanton-corrected string equations of motion. In particular, the BRST operator fails to localize on its large radius representative, which implies that the BRST cohomology of the model as computed in [8] has to be modified. The relevant corrections and deformation theory will be discussed in more detail in [3] from a string field theoretic perspective, where in particular I will show that the obstruction of [6, 7] has a very simple string-field theoretic interpretation. Because the analysis of [3] involves a string field theory expanded around the wrong vacuum, one must be careful not to use the final result when constructing the theory. In particular, it is important to correctly identify the boundary topological metric and give an argument for its independence of Calabi-Yau radius which does not rely on BRST closure and conformal invariance². To avoid circular arguments, we give a non-renormalization theorem which is based on the geometry of the associated moduli spaces. This result assures that the boundary topological metric does not receive instanton corrections, a statement which is important for building the open string field theory [3]. I also give non-renormalization results for the bulk and boundary one-point amplitudes on the disk. Finally, I discuss the relation between string amplitudes and nonlinear sigma model amplitudes with integrated insertions of descendants. Using explicit expressions after localization allows for a proof of their equivalence which does not assume that the background satisfies the string equations of motion. Section 2 reviews the structure of the open-closed A model. Appendix A re-derives localization for the BRST operator in the large radius limit in the Lagrangian framework, while Appendix B discusses the Maslov index of disk instantons.

A word of caution is in order for the mathematically inclined reader. In this paper, I treat the various instanton moduli spaces naively, i.e. I neglect the fact that many of the arguments I use are not rigorous unless one compactifies these spaces appropriately. The mathematical machinery needed for approaching this problem is extremely complex and has recently become available in the book [7], to which I refer the interested reader. To a large extent, the present work and its sequel [3] are an attempt to fill in the holes separating the work of the physics community from that of [6] and [7].

²This is to say that we cannot apply the open string version of the argument of [27], since this assumes that we have correctly identified a conformally invariant string vacuum and the associated BRST charge. Since we only know how to recover this data from string field theory arguments, this would amount to using the conclusion as a hypothesis. In particular, we cannot simply borrow the results of [29].

2 The open topological A-model around a classical background

2.1 Review of the model in the presence of one D-brane

Consider a complex d -dimensional Calabi-Yau manifold X , whose complexified tangent bundle we denote by $\mathcal{T}X = TX \oplus \overline{TX}$, where TX and \overline{TX} are the holomorphic and antiholomorphic tangent bundles. Recall that the nonlinear sigma model associated to X contains worldsheet bosons described by a map ϕ from the worldsheet to X as well as fermions ψ_L, ψ_R , which are sections of the bundles $K_c^{1/2} \otimes \phi^*(\mathcal{T}X) = K_c^{1/2} \otimes \phi^*(TX) \oplus K_c^{1/2} \otimes \phi^*(\overline{TX})$ and $K_a^{1/2} \otimes \phi^*(\mathcal{T}X) = K_a^{1/2} \otimes \phi^*(TX) \oplus K_a^{1/2} \otimes \phi^*(\overline{TX})$. Here K_c, K_a are the canonical and anticanonical line bundles on the worldsheet while $K_c^{1/2}$ and $K_a^{1/2}$ are choices for their square roots, which control the spin structure.

To build the associated A-model, one simply declares these fields to be sections of the ‘twisted’ bundles $\phi^*(TX) \oplus K_c \otimes \phi^*(\overline{TX})$ and $K_a \otimes \phi^*(TX) \oplus \phi^*(\overline{TX})$ [10]. The twisted fields (which we denote by the same symbols) admit decompositions:

$$\psi_L = \chi_L + \lambda_L \quad (1)$$

$$\psi_R = \chi_R + \lambda_R, \quad (2)$$

where χ_L and χ_R are sections of $\phi^*(TX)$ and $\phi^*(\overline{TX})$ and λ_L, λ_R are sections of $K_c \otimes \phi^*(\overline{TX})$ and $K_a \otimes \phi^*(TX)$.

The bulk action of the twisted model (on the disk D) is given by:

$$S = \int_D d^2z \left[\frac{1}{2} g(\partial_z \phi(z, \bar{z}), \partial_{\bar{z}} \phi(z, \bar{z})) + ig(\lambda_L(z, \bar{z}), \partial_{\bar{z}} \chi_L(z, \bar{z})) + ig(\lambda_R(z, \bar{z}), \partial_z \chi_R(z, \bar{z})) \right] + \int_D d^2z R(\chi_L(z, \bar{z}), \lambda_L(z, \bar{z}), \lambda_R(z, \bar{z}), \chi_R(z, \bar{z})), \quad (3)$$

where we wrote $\lambda_L = \lambda_L(z, \bar{z})d_z$, $\lambda_R = \lambda_R(z, \bar{z})d_{\bar{z}}$ etc. Note that λ_L etc need not be holomorphic sections of the corresponding bundles, since we are not imposing the equations of motion.

For later use, we also define:

$$\chi = \chi_L + \chi_R \quad (4)$$

$$\lambda = \lambda_L + \lambda_R, \quad (5)$$

which are sections of $\phi^*(\mathcal{T}X)$ and $K_c \otimes \phi^*(\overline{TX}) \oplus K_a \otimes \phi^*(TX)$, respectively.

The A model admits a nilpotent BRST symmetry, whose action on the fields is given by:

$$\delta_Q \phi = i\xi \chi \quad (6)$$

$$\delta_Q \chi = 0 \quad (7)$$

$$\delta_Q \lambda_L^{\bar{i}} = -\xi \partial \phi^{\bar{i}} - i\xi \chi^{\bar{j}} \Gamma_{\bar{j}\bar{m}}^{\bar{i}} \lambda_L^{\bar{m}} \quad (8)$$

$$\delta_Q \lambda_R^i = -\xi \bar{\partial} \phi^i - i\xi \chi^{\bar{j}} \Gamma_{j\bar{m}}^i \lambda_R^{\bar{m}}, \quad (9)$$

with ξ a Grassmann odd infinitesimal parameter. In these equations, Γ is the Levi-Civita connection on X . As in [10, 8], we define an operator Q through the relation:

$$\delta_Q \Lambda = -i\xi\{Q, \Lambda\} \quad , \quad (10)$$

where Λ is an arbitrary field and $\{Q, \Lambda\}$ stands for the graded commutator.

As pointed out in [8], one can introduce a ‘Floer’ topological D-brane in the background. This is described by a pair (L, A) formed of a Lagrangian cycle L and a connection A living in a complex vector bundle E over L . We let r denote the (complex) rank of E . In the presence of the Floer D-brane (L, A) , the fields ϕ, χ, λ are subject to the boundary conditions:

$$\begin{aligned} \phi(\partial D) &\subset L \\ \partial_n \phi|_{\partial D} &\text{ is a section of } \phi^*(NL) \\ \chi|_{\partial D} &\text{ is a section of } \phi^*(TL) \\ \lambda|_{\partial D} &\text{ is form valued in } \phi^*(TL) \end{aligned} \quad (11)$$

Here we consider the model defined on a disk D , whose boundary we denote by ∂D .

The boundary model is obtained from its bulk counterpart by multiplying the following Wilson loop term to the path integrand:

$$e^{-S_b} := \text{tr} P e^{-\int_{\partial D} \phi^*(A)} \quad (12)$$

In the large radius limit, it was shown in [8] that BRST-invariance of the model requires the connection A to be flat. As mentioned in the introduction, this conclusion will be modified by instanton effects. Therefore, the condition $F_A = 0$ should be viewed as *large radius* string equation of motion, defining a semiclassical background in the open string sector ³.

2.2 Bulk and boundary observables

In this subsection, I recall some well-known results on bulk observables, and re-derive some properties of the boundary observables from a ‘Lagrangian’ point of view.

2.2.1 Bulk observables

The bulk topological observables of the A-model were constructed in [9, 10]. The local observables are Q -closed operators $\mathcal{O}_v(z) = v_{I_1 \dots I_k}(\phi(z)) \chi^{I_1}(z) \dots \chi^{I_k}(z)$ associated with closed forms v on the Calabi-Yau manifold X . Such an operator is large radius BRST-exact (i.e. exact with respect to the BRST operator appropriate in the large radius

³By semiclassical we mean that *worldsheet* quantum effects are being neglected. We shall not treat space-time quantum effects (aka string loop effects) in this paper. To avoid confusion, we prefer to use the term ‘large radius background’ when referring to this approximation.

limit) if and only if the form w is d -exact, where d is the exterior differential on X . Hence the (local) large radius operator BRST cohomology of the bulk is isomorphic with the de Rham cohomology $H^*(X)$ of the target space X . Beyond these operators, one can build nonlocal bulk observables by integrating their descendants. Indeed, it was shown in [27] that bulk observables come in supermultiplets:

$$\phi(\sigma, \theta) = \mathcal{O}(\sigma) + \theta^\alpha \mathcal{O}_\alpha^{(1)}(\sigma) + \theta^\alpha \theta^\beta \mathcal{O}_{\alpha\beta}^{(2)}(\sigma) \quad , \quad (13)$$

with respect to a bulk Q -superspace $(\sigma^\alpha, \theta^\alpha)$ ($\alpha = 1, 2$). Here σ^1, σ^2 denote real coordinates on the worldsheet, related to the complex coordinate z via $z = \sigma^1 + i\sigma^2$. The components of a multiplet are related through the descent equations:

$$d\mathcal{O} = \{Q, \mathcal{O}^{(1)}\} \quad (14)$$

$$d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\} \quad , \quad (15)$$

where we used the worldsheet forms $\mathcal{O}^{(1)} = \mathcal{O}_\alpha^{(1)}(\sigma) d\sigma^\alpha$ and $\mathcal{O}^{(2)} = \mathcal{O}_{\alpha\beta}^{(2)}(\sigma) d\sigma^\alpha \wedge d\sigma^\beta$.

Integrating a two-form descendant over the disk produces a nonlocal observable whose BRST variation is given by⁴:

$$\{Q, \int_D \mathcal{O}^{(2)}\} = \int_{\partial D} \mathcal{O}^{(1)} \quad . \quad (16)$$

The nonlocal observables $\int_D \mathcal{O}^{(2)}$ can be used to construct topological bulk deformations [27, 29], and their insertion in closed sigma model amplitudes produces topological closed string correlators [27, 26, 28]. This is due to the integral over the insertion point which is part of such observables' definition.

2.2.2 Zero-form boundary observables and the boundary BRST operator

Such operators were constructed in [8]. In fact, only the case of a singly-wrapped D-brane (when the bundle E has rank $r = 1$ and A is a $U(1)$ connection) was analyzed explicitly there, through the result for the nonabelian case ($r > 1$) was mentioned without a detailed derivation. For completeness, I give the necessary derivation in Appendix A.

A rank k $End(E)$ -valued form w on L defines a boundary observable on ∂D via:

$$\mathcal{O}_w(x) := w_{\phi(x)}(\chi(x) \dots \chi(x)) \quad , \quad (17)$$

where x is a point of ∂D . Here $w_{\phi(x)}$ is the alternating multilinear form on $T_{\phi(x)}L$ defined by the value of w at the point $\phi(x)$ on L . This expression is geometrically meaningful since $\chi(x)$ are sections of $\phi^*(TL)$, and is generally nonzero even for odd rank k , since χ^I are anti-commuting. Note that \mathcal{O}_w can be viewed as a (Grassmann

⁴The fact that a boundary term appears in the right hand side of this equation was noticed in [11].

even or odd) section of the bundle $\phi^*(\text{End}(E))$. This encodes the natural dependence of boundary observables on Chan-Paton factors.

Let $d_A : \Omega^*(\text{End}(E)) \rightarrow \Omega^{*+1}(\text{End}(E))$ denote the covariant differential⁵ with respect to the connection induced by A on $\text{End}(E)$. It was argued in [8] (and is re-derived by a different method in Appendix A) that the following relation holds in the large radius limit:

$$\{Q_o, \mathcal{O}_w\} = -\mathcal{O}_{d_A w} \quad , \quad (18)$$

where Q_o is the BRST operator in the boundary sector. It follows that topological boundary observables are in one to one correspondence with elements of $H_{d_A}(L, \text{End}(E))$. In particular, the open sector BRST operator can be identified with d_A *in the large radius limit*. The fact that the localized form of Q_o depends on the connection A has far reaching implications for the nature of instanton corrections, which will be explored in more detail in [3]. For the moment, it suffices to note that, since instanton corrections are known to affect the string field theory action [8], the correct string field equations of motion away from the large radius limit will require A to obey an equation of the form:

$$d_A A + \text{instanton corrections} = 0 \quad . \quad (19)$$

In particular, a flat connection A ($F_A = d_A A = 0$) will generally fail to satisfy this equation, which is to say that it defines a background which does not satisfy the string equations of motion. It follows that the large radius form d_A of the localized BRST operator cannot be correct once instanton effects are taken into account. Since the kinetic term of the open string field action is always of the form [1]:

$$S_{kin} = \frac{1}{2} \int \text{tr}(A * Q_o A) \quad , \quad (20)$$

this means that the localized string field action of [8] must contain further corrections induced by worldsheet instanton contributions to Q . As I show in [3] (following standard methods of string field theory, which were already employed for related reasons in [6, 7]), these corrections can be determined indirectly by first building a non-polynomial string field theory around the large radius background given by a flat connection A and then shifting this background to a solution of the corrected string equations of motion. The string field theory action which results after this shift consistently takes into account all of the disk instanton contributions to the dynamics of the model.

2.2.3 One-form boundary observables

Such operators can be built as descendants of boundary zero-form observables. BRST-invariance of the model implies that the boundary operators are components of a Q_o -superspace of coordinates $x \in \partial D$ and θ . This represents the ‘boundary restriction’ of the bulk superspace $(\sigma^\alpha, \theta^\alpha)$. It follows that the components of a boundary superfield

⁵ d_A is a differential (i.e. $d_A^2 = 0$) since A is flat.

$\Psi(x, \theta) = \mathcal{O}(x) + \theta \mathcal{O}^{(1)}(x)$ are related by $d\mathcal{O} = \{Q_o, \mathcal{O}^{(1)}\}$ (the boundary descent equations), where d is the exterior differential along the boundary and $\mathcal{O}^{(1)} = \mathcal{O}^{(1)}(x)dx$. If $\mathcal{O} = \mathcal{O}_w$ is associated with a closed form $w \in \Omega(L, \text{End}(E))$, then it is easy to see that $\mathcal{O}_w^{(1)}(x) = -w_{i_1 \dots i_k}(\phi(x)) \phi^{i_1}(x) \chi^{i_2}(x) \dots \chi^{i_k}(x) dx$. The *trace* of such operators can be integrated along a boundary segment $C \subset \partial D$ to produce a nonlocal observable whose BRST-variation is given by:

$$\{Q_o, \int_C \text{tr}(\mathcal{O}^{(1)})\} = \text{tr}(\mathcal{O}(q)) - \text{tr}(\mathcal{O}(p)) \quad , \quad (21)$$

where p and q are the initial and final points of C .

3 Localization for sigma model open-closed disk amplitudes

We are ready to discuss the geometric formulation of instanton contributions to open-closed amplitudes on the disk. We start by considering sigma model amplitudes, and give largely an off-shell treatment, which does not require BRST-closure of topological operators.

3.1 The amplitudes

We are interested in disk correlators with an arbitrary number of bulk and boundary operator insertions. Consider n forms $v^{(j)}$ ($j = 1..n$) on X and m $\text{End}(E)$ -valued forms $w^{(\alpha)}$ ($\alpha = 0..m-1$) on L . Let k_j and l_α denote their ranks. We shall give a geometric expression for the amplitude:

$$\begin{aligned} & \langle \prod_j \mathcal{O}_{v^{(j)}}(z_j) \prod_\alpha \mathcal{O}_{w^{(\alpha)}}(x_\alpha) \rangle = \\ &= \frac{1}{r} \int \mathcal{D}[\phi, \chi, \lambda] e^{-S_{\text{bulk}}[\phi, \chi, \lambda]} \prod_j v_{i_1 \dots i_{k_j}}^{(j)}(\phi(z_j)) \chi^{i_1}(z_j) \dots \chi^{i_{k_j}}(z_j) \times \\ & \quad \times \text{tr} \prod_\alpha \left(w_{i_1 \dots i_{l_\alpha}}^{(\alpha)}(\phi(x_\alpha)) \chi^{i_1}(x_\alpha) \dots \chi^{i_{l_\alpha}}(x_\alpha) U(x_\alpha, x_{\alpha-1}) \right) . \end{aligned} \quad (22)$$

Here I and i and indices describing tangent directions to X and L , and we order the boundary insertion points $x_0 \dots x_{m-1}$ cyclically on ∂D (figure 1). The symbol $U(x_\alpha, x_{\alpha-1})$ stands for the holonomy operator $Pe^{-\int_{x_{\alpha-1}}^{x_\alpha} \phi^*(A)}$ of $-\phi^*(A)$ along the boundary segment $C_{\alpha-1} = (x_{\alpha-1}, x_\alpha)$ which connects $x_{\alpha-1}$ to x_α in the order given by the orientation of ∂D .

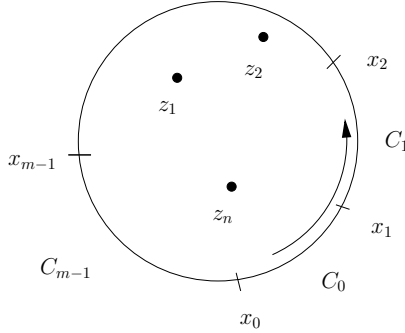


Figure 1. Punctured disk associated to an open-closed tree-level amplitude.

Since some of the observables \mathcal{O} may be Grassmann-odd, we must give a clear ordering prescription for the products appearing in equation (22). Our convention is that the boundary product is taken in the opposite sense of the boundary orientation:

$$\prod_{\alpha} \mathcal{O}_{w(\alpha)}(x_{\alpha}) := \mathcal{O}_{w(m-1)}(x_{m-1}) \dots \mathcal{O}_{w(1)}(x_1) \mathcal{O}_{w(0)}(x_0) \quad , \quad (23)$$

while the product of bulk insertions is taken in the order from n to 1:

$$\prod_j \mathcal{O}_{v(j)}(x_j) = \mathcal{O}_{v(n)}(z_n) \dots \mathcal{O}_{v(2)}(z_2) \mathcal{O}_{v(1)}(z_1) \quad . \quad (24)$$

In (22), we identify x_m with x_0 , i.e. we take the index of boundary insertions to be an element of \mathbb{Z}_m . Hence $U(x_0, x_{-1}) = U(x_0, x_{m-1})$. This amplitude is invariant *up to sign* under an arbitrary permutation of bulk insertions and cyclic permutations of boundary insertions.

3.2 Localization on instanton configurations

As usual in topological field theory, the path integral in (22) localizes on the set of Q -closed configurations, which coincide [8, 9, 10] with with local extrema of the Euclidean action S . This gives a sum of integrals over moduli spaces of instantons. An instanton of our model is a holomorphic map $\phi : D \rightarrow X$ obeying the constraint $\phi(\partial D) \subset L$. The topological type of maps from D to L (which plays the role of ‘instanton class’ and acts as a superselection index) is given by relative homotopy classes $\beta \in \pi_2(X, L)$ ⁶. For each such class, consider the moduli space \mathcal{M}_{β} of all instantons which it contains. This is a non-compact manifold of virtual dimension ⁷ $v_{\beta} = d + \mu(\beta)$, where $\mu : \pi_2(X, L) \rightarrow \mathbb{Z}$

⁶For technical reasons this has to be replaced with $\pi_2^{free} = \pi_0(Map(D, L))$ [20, 7], but I will neglect this here. Also, the instanton moduli spaces considered below may fail to be orientable unless L satisfies extra conditions [7], another issue which will be ignored below.

⁷The virtual dimension is defined as the number of χ zero modes minus the number of λ zero modes, and is given by an index theorem discussed in [6, 7].

is the integer-valued *Maslow index* (whose definition is recalled in Appendix B)⁸. Note that we do *not* mod out by the action of $SL(2, \mathbb{R})$ when building \mathcal{M}_β ⁹.

The value of the Euclidean action S_{bulk} on an instanton configuration ϕ depends only on its relative homotopy class:

$$S_{bulk}(\phi) = \int_D \phi^*(\omega) := S_\beta \quad , \text{ for all } \phi \text{ holomorphic and of class } \beta \quad , \quad (25)$$

where ω is the Kahler class of X . It follows that:

$$\left\langle \prod_j \mathcal{O}_{v(j)}(z_j) \prod_\alpha \mathcal{O}_{w(\alpha)}(x_\alpha) \right\rangle = \sum_{\beta \in \pi_2(X, L)} e^{-S_\beta} I_\beta(x, z; v, w) \quad , \quad (26)$$

with the quantity I_β given by an integral of

$$\prod_j v_{I_1 \dots I_{k_j}}^{(j)}(\phi(z_j)) \chi^{I_1}(z_j) \dots \chi^{I_{k_j}}(z_j) \text{tr} \prod_\alpha \left(w_{i_1 \dots i_{l_\alpha}}^{(\alpha)}(\phi(x_\alpha)) \chi^{i_1}(x_\alpha) \dots \chi^{i_{l_\alpha}}(x_\alpha) U(x_\alpha, x_{\alpha-1}) \right) \quad (27)$$

over the supermanifold associated to \mathcal{M}_β . The fermionic coordinates on this supermanifold are the Q -partners of the infinitesimal variations of a holomorphic map ϕ . Since functions over this superspace can be identified with forms over \mathcal{M}_β , it follows that I_β reduces to the integral of a certain form over \mathcal{M}_β .

3.3 The universal instanton formalism

To formulate localization for I_β , it is convenient to use a formalism similar to that discussed in [9, 10]. Let us consider the *universal instanton* $\Phi : D \times \mathcal{M}_\beta \rightarrow X$, defined by $\Phi(z, t) := \phi_t(z)$, where ϕ_t is the instanton of modulus $t \in \mathcal{M}_\beta$. In this description, we view \mathcal{M}_β as a parameter space for instantons of class β . We will often identify t with ϕ_t , thereby writing $\Phi(z, \phi) = \phi(z)$. We also consider the restriction $\Phi_\partial := \Phi|_{\partial D \times \mathcal{M}_\beta} : \partial D \times \mathcal{M}_\beta \rightarrow L$.

The pair of maps (Φ, Φ_∂) allows us to pull-back forms on X and $End(E)$ -valued forms on L to the moduli space \mathcal{M}_β . We analyze this procedure for the bulk and boundary sectors in turn.

3.3.1 The bulk sector

Consider a form $v \in \Omega_k(X)$ and its pull-back $V := \Phi^*(v) \in \Omega_k(D \times \mathcal{M}_\beta)$. The latter decomposes as $V = V_0 + V_1 + V_2$, with $V_j \in \Omega_j(D) \otimes \Omega_{k-j}(\mathcal{M}_\beta)$. For each point z in

⁸It can be shown that the Maslow index $\mu(\beta)$ is zero for all β provided that the cycle L is *special* Lagrangian (the *special* condition means that the restriction to L of the imaginary part of the holomorphic 3-form of X vanishes). However, the open-closed topological A model makes perfect sense without this restriction, and working with general Lagrangian cycles is important if one wishes to study non-BPS D-branes.

⁹In this respect, we follow the same convention as in [9, 10] and [8] for topological *sigma model* amplitudes. When discussing topological *string* amplitudes, we will of course *have to* divide by this group.

the interior of D , define an evaluation map $e_z : \mathcal{M}_\beta \rightarrow X$ by:

$$e_z(\phi) := \phi(z) \quad . \quad (28)$$

Identifying $\{z\} \times \mathcal{M}_\beta$ with \mathcal{M}_β , this can be expressed as composition of Φ with j_z , where $j_z : \{z\} \times \mathcal{M}_\beta \rightarrow D \times \mathcal{M}_\beta$ is the inclusion map of the set $\{z\} \times \mathcal{M}_\beta$ in $D \times \mathcal{M}_\beta$. We are interested in the form $V_z := (e_z)^*(v) = j_z^*(V) = V_0(z) \in \Omega_k(\mathcal{M}_\beta)$.

Now let us consider the case when v is closed. Since exterior differentiation on $D \times \mathcal{M}_\beta$ decomposes as $d = d_D + d_{\mathcal{M}}$, with d_D and $d_{\mathcal{M}}$ the exterior differentials on D and \mathcal{M}_β , the property $d\Phi^*(V) = \Phi^*(dv) = 0$ implies the geometric descent equations:

$$\begin{aligned} d_D V_2 &= 0 \\ d_D V_1 &= -d_{\mathcal{M}} V_2 \\ d_D V_0 &= -d_{\mathcal{M}} V_1 \quad . \end{aligned} \quad (29)$$

In this case, the z -dependence of V_z is exact, i.e. $V_{z'} - V_z$ is an exact form on \mathcal{M}_β . This follows from the third descent equation, upon integration along a curve $\gamma \subset \text{Int} D$ connecting z and z' :

$$V_{z'} - V_z = -d_{\mathcal{M}} \int_\gamma V_1 \quad . \quad (30)$$

3.3.2 The boundary sector

Consider a rank l form w on L with values in the bundle $\text{End}(E)$. The map Φ_∂ allows us to define the pulled-back bundle $\mathcal{E} := (\Phi_\partial)^*(E)$ and the pulled-back form $W := (\Phi_\partial)^*(w) \in \Omega_l(\partial D \times \mathcal{M}_\beta, \text{End}(\mathcal{E}))$. Since E is flat, \mathcal{E} is a flat vector bundle over $\partial D \times \mathcal{M}_\beta$. In particular, the connection A on E induces a flat connection \mathcal{A} on \mathcal{E} , and a covariant differential $d_{\mathcal{A}}$ on $\text{End}(\mathcal{E})$. Using the inclusion map j_x of $\{x\} \times \mathcal{M}_\beta$ into $\partial D \times \mathcal{M}_\beta$, we can define ‘restricted bundles’ $\mathcal{E}(x) := j_x^*(\mathcal{E})$ on \mathcal{M}_β , for all points x on the boundary of the disk, and evaluation maps $e_x := \Phi \circ j_x : \mathcal{M}_\beta \rightarrow L$. The bundles $\mathcal{E}(x)$ inherit flat connections \mathcal{A}_x induced by restricting \mathcal{A} .

Since the bundle \mathcal{E} is flat, the holonomy of A defines maps $\mathcal{U}((x', t'), (x, t))$ from the fiber $\mathcal{E}|_{(x, t)}$ to the fiber $\mathcal{E}|_{(x', t')}$ for all x, x' on the boundary of D and all t, t' in \mathcal{M}_β . These maps depend only on the pairs (x, t) and (x', t') . In particular, we have isomorphisms $\mathcal{U}(x', x)(t) = \mathcal{U}((x', t), (x, t))$ between $\mathcal{E}(x)|_t$ and $\mathcal{E}(x')|_t$ (see figure 2). These give a bundle isomorphism $U(x', x)$ between $\mathcal{E}(x)$ and $\mathcal{E}(x')$. These identifications depend on the flat connection A on the topological D-brane. For our correlator, we are interested in forms of the type $W_x := (e_x)^*(w) = (j_x)^*(W) \in \Omega_l(\mathcal{M}_\beta, \text{End}(\mathcal{E}(x)))$.

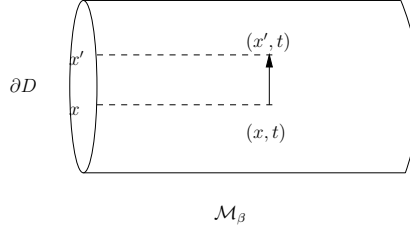


Figure 2. Two points (x, t) and (x', t) in the direct product $\partial D \times \mathcal{M}_\beta$. Parallel transport along the oriented path shown in the figure defines the holonomy operator $\mathcal{U}(x, x')(t)$.

Now consider the case when w is d_A -closed on L . Then the pulled-back form W is d_A -closed on $\partial D \times \mathcal{M}_\beta$, and W_x is $d_{\mathcal{A}_x}$ -closed on \mathcal{M}_β . Moreover, W decomposes as:

$$W = W_0 + W_1 \quad , \quad (31)$$

where W_0 and W_1 are zero- and one-forms in the ∂D directions. In particular, we have $W_x = W_0(x)$. One can similarly decompose the connection one-form \mathcal{A} :

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1. \quad (32)$$

It is the easy to see that the flatness condition $d_A \mathcal{A} = 0$ implies that \mathcal{A}_0 and \mathcal{A}_1 are (pull-backs of) forms on \mathcal{M}_β and ∂D , respectively (i.e. they have no dependence of the ∂D and \mathcal{M}_β directions), and that \mathcal{A}_0 and \mathcal{A}_1 are flat on \mathcal{M}_β and ∂D . The condition $d_A W = 0$ implies the geometric boundary descent equations:

$$\begin{aligned} d_{\mathcal{A}_1} W_1 &= 0 \\ d_{\mathcal{A}_1} W_0 &= -d_{\mathcal{A}_0} W_1 \end{aligned} \quad (33)$$

$$d_{\mathcal{A}_0} W_0 = 0 \quad . \quad (34)$$

3.4 Localization formula for I_β

With these preliminaries, we are ready to give the localization formula for I_β :

$$I_\beta = \frac{1}{r} \int_{\mathcal{M}_\beta} V_{z_n}^{(n)} \wedge \dots \wedge V_{z_1}^{(1)} \text{tr} \left[W_{x_{m-1}}^{(m-1)} \mathcal{U}(x_{m-1}, x_{m-2}) W_{x_{m-2}}^{(m-2)} \mathcal{U}(x_{m-2}, x_{m-3}) W_{x_{m-3}}^{(m-3)} \dots \mathcal{U}(x_1, x_0) W_{x_0}^{(0)} \mathcal{U}(x_0, x_{m-1}) \right] , \quad (35)$$

where r denotes the rank of E . This relation can be justified though path integral arguments similar to those of [9, 8].

3.5 Ghost grading and the ghost number anomaly

The open-closed nonlinear sigma model has a global $U(1)$ ‘ghost number’ symmetry under which ϕ, χ and λ have charges 0, +1 and -1 respectively. In particular, the

bulk and boundary local observables \mathcal{O}_v , \mathcal{O}_w have charges $rank v$ and $rank w$. In the quantum theory, this symmetry is anomalous due to the nontrivial geometric character of the Grassmann-odd worldsheet fields. In a bosonic background configuration ϕ , the anomaly is computed by the index of the associated Dirac operator, which is proportional to the difference between the number of χ zero modes and the number of λ zero modes. This follows by the standard argument of [9]. The result is given by an index theorem [6, 7], on surfaces with boundaries which for the case of disks gives the virtual dimension $d + \mu(\beta)$ mentioned above. The contribution $d = \frac{1}{2}dim_{\mathbb{R}} X$ is the standard ghost number anomaly on the disk, while $\mu(\beta)$ appears as a correction due to the nontrivial boundary conditions on the fermions.

Knowledge of the ghost number anomaly allows us to write down the selection rules for amplitudes. Namely, the instanton contribution I_β vanishes unless:

$$\sum_j rank v_j + \sum_\alpha rank w_\alpha = d + \mu(\beta) \quad . \quad (36)$$

This follows immediately from ghost number conservation or more directly from the localization formula (35).

3.6 Exceptional sigma model amplitudes

Let us take a closer look at the disk amplitudes containing one or two boundary insertions or a single bulk insertion. These have certain non-renormalization properties, which originate from the fact that specifying the position of such sets of punctures does not suffice to completely fix the gauge under the the $SL(2, \mathbb{R})$ symmetry of the disk.

3.6.1 The two-point boundary amplitude

This is the sigma model correlator $\langle \mathcal{O}_{w(1)}(x_1) \mathcal{O}_{w(0)}(x_0) \rangle = \sum_\beta I_\beta e^{-S_\beta}$, with:

$$I_\beta = \frac{1}{r} \int_{\mathcal{M}_\beta} tr \left[W_{x_1}^{(1)} \mathcal{U}(x_1, x_0) W_{x_0}^{(0)} \mathcal{U}(x_0, x_1) \right] \quad . \quad (37)$$

It is easy to see that the obvious action of $SL(2, \mathbb{R})$ has a one-dimensional subgroup G which fixes the points x_0 and x_1 . This is isomorphic with the additive group \mathbb{R} of real numbers. If one maps the disk to the upper half plane such that x_0 and x_1 are mapped to the origin 0 and the unit 1, then the elements of G are $SL(2, \mathbb{R})$ matrices of the form:

$$A_\xi = \begin{bmatrix} e^\xi & 0 \\ 2sinh(\xi) & e^{-\xi} \end{bmatrix} \quad , \quad (38)$$

which obey $A_{\xi_1} A_{\xi_2} = A_{\xi_1 + \xi_2}$ and $A_0 = Id$. In this presentation, an element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$ acts on the upper half plane via $z \rightarrow f_A(z) = \frac{az+b}{cz+d}$. For ease of notation, we denote f_{A_ξ} simply by f_ξ .

The transformations f_ξ act naturally on the moduli space \mathcal{M}_β . If ϕ is an instanton in \mathcal{M}_β , then the action of A_ξ takes it into $\phi \circ f_{-\xi}$. The composite map is still an instanton, since f_ξ is holomorphic. We denote the resulting action of \mathbb{R} on \mathcal{M}_β by ρ , i.e. we write $\rho_\xi(\phi) := \phi \circ f_{-\xi}$. Since f_ξ fixes the points x_0 and x_1 , such transformations of \mathcal{M}_β leave the evaluation maps unchanged:

$$e_{x_\alpha} \circ \rho_\xi = e_{x_\alpha} \quad \text{for } \alpha = 0, 1 \quad . \quad (39)$$

Indeed, one has $e_{x_\alpha}(\rho_\xi(\phi)) = e_{x_\alpha}(\phi \circ f_{-\xi}) = (\phi \circ f_{-\xi})(x_\alpha) = \phi(f_{-\xi}(x_\alpha)) = \phi(x_\alpha) = e_{x_\alpha}(\phi)$. This invariance property implies that e_{x_α} descend to evaluation maps \hat{e}_{x_α} defined on the quotient $\hat{\mathcal{M}}_\beta = \mathcal{M}_\beta/\mathbb{R}$. If $\pi : \mathcal{M}_\beta \rightarrow \hat{\mathcal{M}}_\beta$ is the associated projection, the induced maps satisfy $e_{x_\alpha} = \hat{e}_{x_\alpha} \circ \pi$, which shows that the forms $W_{x_\alpha}^{(\alpha)} = e_{x_\alpha}^*(w^{(\alpha)})$ are basic on this fibration:

$$W_{x_\alpha}^{(\alpha)} = \pi^*(\omega_{x_\alpha}^{(\alpha)}) \quad , \quad (40)$$

with $\omega_{x_\alpha}^{(\alpha)} = \hat{e}_{x_\alpha}^*(w^{(\alpha)})$ some forms on $\hat{\mathcal{M}}_\beta$ taking values in the bundles $End(\hat{\mathcal{E}}_{x_\alpha})$, where $\hat{\mathcal{E}}_{x_\alpha} := \hat{e}_{x_\alpha}^*(E)$.

We can now prove a non-renormalization theorem for the two-point boundary amplitude. For this, consider the case when the homotopy class β is nontrivial. In this situation, the action ρ on \mathcal{M}_β reduces its dimension, i.e. the quotient $\hat{\mathcal{M}}_\beta$ has dimension one unit less than \mathcal{M}_β . It immediately follows that the integral (37) must vanish¹⁰. The case $\beta = 0$ (the trivial homotopy class) is different. In this situation, the space \mathcal{M}_0 coincides with L , since area-minimizing maps of trivial relative homotopy must be constant on the disk D . Since the action ρ is trivial on such maps, the quotient $\hat{\mathcal{M}}_0$ coincides with \mathcal{M}_0 and the argument given above can no longer be applied. It follows that the entire contribution to the two-point amplitude comes from the trivial instanton sector, and can be expressed geometrically as an integral over L :

$$\langle \mathcal{O}_{w^{(1)}} \mathcal{O}_{w^{(0)}} \rangle = I_0 = \frac{1}{r} \int_L tr \left[w^{(1)} \wedge w^{(0)} \right] \quad . \quad (41)$$

Since this quantity localizes exclusively on the trivial instanton sector, it does not depend on the insertion points x_0 and x_1 .

The two-point *sigma model* correlator on the disk defines the so-called *boundary topological metric*:

$$\rho(w_1, w_2) := \langle \mathcal{O}_{w_1} \mathcal{O}_{w_2} \rangle \quad . \quad (42)$$

Hence we recover the result that *the boundary topological metric does not receive instanton corrections*. A different argument to the same effect can be extracted from [29].

However, the approach of [29] assumes a background which is conformally invariant

¹⁰When performing this integral one has to evaluate the integrand on bases $u_1..u_D$ of tangent vectors to \mathcal{M}_β , where $D = \dim \mathcal{M}_\beta$; this has to be done on a coordinate cover of \mathcal{M}_β subordinate to a partition of unity, according to the definition of integration of forms. One can always assume that u_1 is in the direction of the \mathbb{R} fiber, which implies that $\pi_*(u_1) = 0$. Since both forms involved in (37) are basic, their values on u_j can be expressed in terms of $\pi_*(u_j)$, hence the value of the integrand on $u_1..u_D$ is always zero.

and satisfies the equations of motion, which is precisely the assumption we wish to avoid.

3.6.2 The one-point bulk amplitude on the disk

A similar non-renormalization result can be derived for disk amplitudes $\langle \mathcal{O}_v(z) \rangle$ with a single bulk insertion. In this case, the stabilizer of the insertion point z in $SL(2, \mathbb{R})$ is isomorphic with the rotation group $SO(2)$. Mapping the disk to the upper half-plane such that z is mapped into the imaginary unit, the $SL(2, \mathbb{R})$ matrices which stabilize i have the form:

$$A_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} , \quad (43)$$

with $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$. As before, we denote their action on the disk by $f_\theta = f_{A_\theta}$. One has the expansion $\langle \mathcal{O}_v(z) \rangle = \sum_\beta I_\beta e^{-S_\beta}$, with:

$$I_\beta = \frac{1}{r} \int_{\mathcal{M}_\beta} H_A V_z , \quad (44)$$

where H_A is a function on \mathcal{M}_β given by the fiber-wise trace of the holonomy operator $\mathcal{U}(x-, x+) : \mathcal{E}(x) \rightarrow \mathcal{E}(x)$ associated to the path which connects a point $x \in \partial D$ with itself after winding once around the boundary of the disk (figure 3). The fiber-wise trace H_A does not depend on the choice of x , since changing x to $x' \in \partial D$ amounts to a similarity transformation¹¹:

$$\mathcal{U}(x'-, x'+) = \mathcal{U}(x', x) \mathcal{U}(x-, x+) \mathcal{U}(x', x)^{-1} . \quad (45)$$

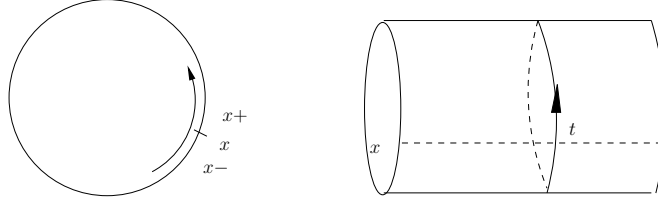


Figure 3. Parallel around the disk boundary induces an automorphism of the bundle $\mathcal{E}(x)$ defined over \mathcal{M}_β .

Now consider the action $\rho_\theta(\phi) := \phi \circ f_{-\theta}$ of $SO(2)$ on \mathcal{M}_β and notice as before that it preserves the evaluation map at z :

$$e_z \circ \rho_\theta = e_z . \quad (46)$$

¹¹One can in fact identify $H_A(\phi)$ with the spacetime holonomy $W(\phi) = \frac{1}{r} \text{tr} P e^{-\int_{\partial D} \phi^*(A)}$, which depends only on the homotopy class β ; this allows one to write $I_\beta = W_\beta \int_{\mathcal{M}_\beta} V_z$. Since I_β will localize on the trivial instanton sector, this turns out to be irrelevant.

Hence e_z descends to a map \hat{e}_z defined on the quotient $\hat{\mathcal{M}}_\beta = \mathcal{M}_\beta/SO(2)$, which satisfies $e_z = \hat{e}_z \circ \pi$ with respect to the associated projection π . This implies that the form V_z is basic:

$$V_z = \pi^*(\omega) \quad , \quad (47)$$

with ω a form on $\hat{\mathcal{M}}_\beta$. As in the previous subsection, we conclude that $I_\beta = 0$ unless β is the trivial homotopy class. Hence the bulk one-point functions on the disk are given exactly by their large radius (trivial instanton sector) expression:

$$\langle \mathcal{O}_v \rangle = \int_L v \quad . \quad (48)$$

To obtain the last formula, we used that fact that $\mathcal{M}_0 = L$ and noticed that the holonomy of E around a curve on L which is homotopically trivial in L is given by the identity operator¹².

3.6.3 The one-point boundary amplitude on the disk

Finally, we consider one-point boundary amplitudes $\langle \mathcal{O}_w(x) \rangle$. A similar argument shows that such correlators do not receive instanton corrections, and hence are given by their large radius value:

$$\langle \mathcal{O}_w \rangle = \frac{1}{r} \int_L tr(w) \quad . \quad (49)$$

This allows us to express them in terms of the boundary topological metric:

$$\langle \mathcal{O}_w \rangle = \rho(w, 1_E) = \langle \mathcal{O}_w \mathcal{O}_{1_E} \rangle \quad , \quad (50)$$

where 1_E is the identity endomorphism of E .

4 Open-closed string amplitudes on the disk

4.1 Preliminary remarks

A complete study of the open-closed string theory based on the topological A-model requires that we consider its coupling to the open-closed version of two-dimensional topological gravity [30]. Though the present paper does not discuss open-closed topological gravity, it is not hard to understand the final effect of the construction. As in the closed string case, coupling to topological gravity plays the role of implementing integration over worldsheet metrics. In fact, the resulting model contains the

¹²What we are considering here is a coupling of the A-model boundary state to bulk *A-model* operators. One should not confuse such correlators with the couplings of the boundary A-model vacuum to *B-model* observables. The latter have a very different character and are, in particular, responsible for the mass of wrapped D-branes [22, 23, 24, 25]. In particular, our non-renormalization arguments say nothing about such couplings.

two-dimensional metric as a dynamical variable, and diffeomorphism invariance is recovered only after integrating over all such metrics. As in the closed string case, this can be formally encoded in the statement that bulk and boundary sigma model observables are promoted to string observables upon multiplication with puncture operators. The latter are local operators of topological gravity, associated to punctures on a given Riemann surface. In the open-closed case, these operators come in bulk and boundary incarnations, which we denote by $P(z)$ and $P(x)$. Given bulk and boundary sigma model observables $\mathcal{O}_v(z)$ and $\mathcal{O}_w(x)$, the associated string operators are:

$$\hat{\mathcal{O}}_v(z) = \mathcal{O}_v(z)P(z) \quad (51)$$

$$\hat{\mathcal{O}}_w(x) = \mathcal{O}_w(x)P(x) \quad , \quad (52)$$

and we are interested in amplitudes containing such insertions:

$$\langle \langle \prod_j \mathcal{O}_{v(j)} \prod_\alpha \mathcal{O}_{w(\alpha)} \rangle \rangle := \langle \prod_j \hat{\mathcal{O}}_{v(j)} \prod_\alpha \hat{\mathcal{O}}_{w(\alpha)} \rangle_{string} \quad . \quad (53)$$

Inclusion of $P(z)$ and $P(x)$ is necessary for a consistent implementation of the integral over puncture positions.

In fact, this description is somewhat simplified, since it assumes that we have chosen the ‘puncture representation’ of open-closed Riemann surfaces. Following standard procedures of string field theory [15, 16], one can work with a more general description, which uses a different parameterization of the associated moduli space. It is well-known from the operator (Segal) formalism of conformal field theories [34, 35] that the geometric objects necessary for a correct description of conformal field theory are punctured open-closed Riemann surfaces together with a choice of holomorphic coordinates around the punctures. Such objects are elements of an infinite-dimensional moduli space, which projects onto the moduli space of uncoordinatized punctured surfaces through the map which forgets the coordinates around all punctures. The correct description of Polyakov amplitudes is through integrals of certain differential forms over finite-dimensional chains defined on this infinite-dimensional moduli space. The precise choice of these chains amounts to a specification of string vertices, and is constrained by certain consistency conditions (the geometric vertex equations of [15, 16]) which express the requirement that the domain of integration in string perturbation theory gives a single-cover of the moduli space of uncoordinatized punctured surfaces. The ‘puncture representation’ arises from a particular choice of open-closed string vertices, which consists of taking them to carry infinite-length stubs and strips. This is related [19] to the conformal normal ordering prescription of [18] used in the traditional description of string perturbation theory.

The purpose of recalling these well-known results is to stress that all string amplitudes considered in this paper are taken in the puncture representation, hence our string vertices carry infinite length stubs and strips associated with each open and closed string insertion. This is by no means the most general consistent approach, and other choices are more suitable for foundational studies or for the rigorous definition of the associated string measure.

Ability to write Polyakov amplitudes as integrals over a moduli space of complex Riemann surfaces (i.e. as integrals over conformal equivalence classes of two-dimensional metrics) depends on the assumption that the underlying sigma model is *off-shell* conformally invariant, i.e. is a ‘conformal topological field theory’ in the sense of [27]. Many of the standard geometric manipulations of string perturbation theory are justified only in the presence of off-shell conformality. In particular, the equivalence of amplitudes defined by two conformally equivalent Riemann surfaces is not correct unless off-shell conformality holds—for example, there is no way to identify the amplitudes on a surface with finite size open or closed string boundaries with the amplitude defined on the conformally equivalent punctured surface (which can be visualized as carrying infinitely long stubs and strips at the insertions). Much of the geometric discussion of localized string amplitudes presented below can be carried out without the off-shell conformality assumption, but we wish to warn the reader that this is a mathematical artifact due to the fact that we tacitly use the puncture representation when writing all string localization formulae¹³. One must assume off-shell conformal invariance if one wishes to interpret these objects as standard string amplitudes obeying the axiomatic requirements of [16].

Let me add a few observations on open-closed topological gravity. As in the closed case, pure topological gravity on open-closed Riemann surfaces contains many more observables beyond the puncture operators $P(z)$ and $P(x)$. These can be used to define bulk and boundary *gravitational* descendants of our ‘primary’ operators $\hat{O}_v(z)$ and $\hat{O}_w(x)$. In particular, one could use field theory arguments in order to propose recursion relations between string amplitudes of primaries (to be discussed below) and amplitudes containing gravitational descendants (which will not be discussed in this paper). In the closed case, pure topological gravity and topological gravity coupled to $c < 1$ matter are integrable models, related to matrix models and KdV hierarchies [21]. It is an interesting question to what extent such results generalize to the open-closed case. To my knowledge, little work has been done in this direction.

4.2 Moduli spaces of punctured disks

The localization formula for string amplitudes involves integrals over a moduli space of instantons associated with punctured disks. To define these, consider the unit disk D together with n bulk points z_j and m boundary points x_α , as in Subsection 2.1. Mapping the disk to the upper half-plane (such that ∂D is mapped into the real axis), we have $SL(2, \mathbb{R})$ transformations $z \rightarrow f_A(z) = \frac{az+b}{cz+d}$ induced by real matrices

¹³No such restriction is needed for *sigma model* amplitudes, which make perfect physical sense without the conformality assumption. In fact, one could consider the open-closed sigma model on almost Kahler manifolds, by performing the boundary extension of the closed model of [9]. Though in this paper we consider the Calabi-Yau model for simplicity (which assures off-shell conformal invariance of the sigma model in the *closed* sector), there is no need to impose conformal invariance when discussing open-closed sigma model amplitudes, and we have avoided doing so in previous sections. Conformal invariance in the open sector can be assured by restricting to special Lagrangian cycles L .

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of unit determinant. Such transformations preserve the orientation of ∂D , since the function f_A is strictly increasing when restricted to the real axis. In particular, the $SL(2, \mathbb{R})$ action preserves the cyclic ordering of the points x_α around the boundary.

We define a moduli space $\mathcal{M}(\beta) = \mathcal{M}_m^n(\beta)$ of maps ϕ associated with bulk-boundary punctured disks. More precisely, since the un-punctured disk D has no moduli, the moduli space of punctured disks with n bulk and m boundary punctures coincides with the moduli space of bulk and boundary configurations $z = (z_1 \dots z_n) \in (Int D)^n$ and $x = (x_0 \dots x_{m-1}) \in (\partial D)_{cycl}^n$, where we consider only boundary configurations which preserve the cyclic order on ∂D . This moduli space can be described as the quotient of the set of triples (z, x, ϕ) via the obvious action of $SL(2, \mathbb{R})$:

$$\begin{aligned} z &= (z_1 \dots z_n) \rightarrow (f_A(z_1) \dots f_A(z_n)) \\ x &= (x_0 \dots x_{m-1}) \rightarrow (f_A(x_0) \dots f_A(x_{m-1})) \\ \phi &\rightarrow \phi \circ f_A^{-1} . \end{aligned} \tag{54}$$

Since we ‘integrate’ over puncture positions (and divide by $SL(2, \mathbb{R})$), the real dimension of $\mathcal{M}(\beta)$ equals $d + 2n + m - 3 + \mu(\beta)$ (for the moment we exclude cases with $n + m < 3$). It will prove useful to also consider the moduli space $\tilde{\mathcal{M}}(\beta) = \tilde{\mathcal{M}}_m^n(\beta)$ of triples (z, x, ϕ) *before* performing the $SL(2, \mathbb{R})$ quotient. This is a direct product:

$$\tilde{\mathcal{M}}(\beta) = S_n \times C_m \times \mathcal{M}_\beta , \tag{55}$$

where $S_n = \{(z_1 \dots z_n) \in (Int D)^n \mid z_1 \dots z_n \text{ are all distinct}\}$ and $C_m := \{(x_0 \dots x_{m-1}) \in (\partial D)^m \mid x_0 \dots x_{m-1} \text{ are all distinct and cyclically ordered along } \partial D\}$. The moduli space $\mathcal{M}(\beta)$ is the quotient of $\tilde{\mathcal{M}}(\beta)$ through the $SL(2, \mathbb{R})$ action (54).

Finally, we consider bulk and boundary evaluation maps $\tilde{e}_j : \tilde{\mathcal{M}}(\beta) \rightarrow X$ and $\tilde{e}_\alpha : \tilde{\mathcal{M}}(\beta) \rightarrow L$ defined through:

$$\tilde{e}_j(z, x, \phi) := \phi(z_j) \tag{56}$$

$$\tilde{e}_\alpha(z, x, \phi) := \phi(x_\alpha) . \tag{57}$$

These are obviously invariant under the action (54). Hence they induce well-defined maps $e_j : \mathcal{M}(\beta) \rightarrow X$ and $e_\alpha : \mathcal{M}(\beta) \rightarrow L$ on the quotient space. If we let $p : \tilde{\mathcal{M}}(\beta) \rightarrow \mathcal{M}(\beta)$ denote the obvious projection, then we have:

$$\tilde{e}_j = e_j \circ p , \quad \tilde{e}_\alpha = e_\alpha \circ p . \tag{58}$$

The maps $e_\alpha, \tilde{e}_\alpha$ allow us to define pulled-back bundles $\mathcal{E}_\alpha := e_\alpha^*(E)$, $\tilde{\mathcal{E}}_\alpha = \tilde{e}_\alpha^*(E)$ on the moduli spaces $\mathcal{M}(\beta)$ and $\tilde{\mathcal{M}}(\beta)$. If $u = (z, x, \phi)$ is a point in $\tilde{\mathcal{M}}(\beta)$, the fiber of $\tilde{\mathcal{E}}_\alpha$ at u coincides with the fiber of E at $\phi(x_\alpha)$. The flat connection on E defines parallel transport operators $U(\phi(x_{\alpha_2}), \phi(x_{\alpha_1}))$ from $E_{\phi(x_{\alpha_1})}$ to $E_{\phi(x_{\alpha_2})}$, which induce bundle isomorphisms $\tilde{\mathcal{U}}_{\alpha_2 \alpha_1}$ from $\tilde{\mathcal{E}}_{\alpha_1}$ to $\tilde{\mathcal{E}}_{\alpha_2}$. Equations (58) imply that the bundles $\tilde{\mathcal{E}}_\alpha$

are pull-backs through the projection p of the bundles \mathcal{E}_α . In fact, it is easy to see that the maps $\tilde{\mathcal{U}}_{\alpha_2\alpha_1}$ are also well-behaved with respect to this projection, i.e. they descend to isomorphisms $\mathcal{U}_{\alpha_2\alpha_1}$ from \mathcal{E}_{α_1} to \mathcal{E}_{α_2} . The latter give us a way to identify the various bundles \mathcal{E}_α , which will be crucial when writing string amplitudes. The identifications $\mathcal{U}_{\alpha_2\alpha_1}$ depend on the flat connection A on the cycle L .

4.3 Localization formula for string amplitudes

We consider the (disk) string amplitude $\mathcal{B} = \langle \prod_j \mathcal{O}_{v^{(j)}}(z_j) \prod_\alpha \mathcal{O}_{w^{(\alpha)}}(x_\alpha) \rangle$, where $v^{(j)}$, $w^{(\alpha)}$ are forms on X and L as in Subsection 2.1.1. Since we are interested in working off-shell, we do *not* require these forms to be closed. The topological character of the model implies that such amplitudes localize on disk instantons. Hence one has an expansion:

$$\mathcal{B}(z, x; v, w) = \sum_{\beta \in \pi_2(X, L)} e^{-S_\beta} J_\beta(z, x, v, w) \quad . \quad (59)$$

We claim that the localization formula for J_β is:

$$J_\beta = \frac{1}{r} \int_{\mathcal{M}(\beta)} \left[\wedge_{j=n}^1 e_j^*(v^{(j)}) \right] \wedge tr \left[\wedge_{\alpha=m-1}^0 \left(e_\alpha^*(w^{(\alpha)}) \mathcal{U}_{\alpha, \alpha-1} \right) \right] \quad , \quad (60)$$

where $\mathcal{M}(\beta)$ and e_j, e_α are the moduli spaces and bulk/boundary evaluation maps defined in the previous subsection. In this equation, $e_j^*(v^{(j)})$ and $e_\alpha^*(w^{(\alpha)})$ are complex-valued, respectively $End(\mathcal{E}_\alpha)$ -valued forms on \mathcal{M}_β , while tr denotes the fiber-wise trace. Equation (60) can be justified through a standard analysis of path integral localization.

Let us write the ghost number selection rule for J_β :

$$\sum_{j=1}^n rank v_j + \sum_{\alpha=0}^{m-1} rank w_\alpha = dim \mathcal{M}_m^n(\beta) = d + 2n + m - 3 + \mu(\beta) \quad . \quad (61)$$

In the conformal case we have $\mu(\beta) = 0$ for all β , which gives a global selection rule for the disk string amplitude \mathcal{B} :

$$\sum_{j=1}^n rank v_j + \sum_{\alpha=0}^{m-1} rank w_\alpha = d + 2n + m - 3 \quad . \quad (62)$$

4.4 Relation of string amplitudes to sigma model amplitudes of descendants

We would like to give an open-closed version of the closed string argument [28] that nonlinear sigma model amplitudes with an appropriate number of integrated descendants reproduce string amplitudes. More precisely, we wish to show that the instanton contribution J_β to the string amplitude \mathcal{B} has one the following alternate expressions:

$$\begin{aligned}
J_\beta &= \int_{S_{n-1}(z_1) \times C_{m-1}(x_0)} \int^{(\beta)} \mathcal{D}[\phi, \chi, \lambda] \left[\wedge_{j=n}^2 \mathcal{O}_{v^{(j)}}^{(2)} \right] \mathcal{O}_{v^{(1)}}(z_1) \\
&\wedge \operatorname{tr} \left(\wedge_{\alpha=m-1}^1 \left[\mathcal{O}_{w^{(\alpha)}}^{(1)}(x_\alpha) U(x_\alpha, x_{\alpha-1}) \right] \mathcal{O}_{w^{(0)}}(x_0) U(x_0, x_{m-1}) \right) = \\
&= \int_{S_{n-1}(z_1) \times C_{m-1}(x_0)} \int_{\mathcal{M}_\beta} \left[\wedge_{j=n}^2 V_2^{(j)} \right] \wedge V_0^{(1)}(z_1) \wedge \operatorname{tr} \left(\wedge_{\alpha=m-1}^1 \left[W_1^{(\alpha)} \mathcal{U}(x_\alpha, x_{\alpha-1}) \right] \wedge W_0^{(0)}(x_0) \mathcal{U}(x_0, x_{m-1}) \right) ,
\end{aligned} \tag{63}$$

$$\begin{aligned}
J_\beta &= \int_{S_n \times C_{m-3}(x_0, x_1, x_2)} \int^{(\beta)} \mathcal{D}[\phi, \chi, \lambda] \left[\wedge_{j=n}^1 \mathcal{O}_{v^{(j)}}^{(2)} \right] \\
&\wedge \operatorname{tr} \left(\wedge_{\alpha=m-1}^3 \left[\mathcal{O}_{w^{(\alpha)}}^{(1)}(x_\alpha) U(x_\alpha, x_{\alpha-1}) \right] \mathcal{O}_{w^{(2)}}(x_2) U(x_2, x_1) \mathcal{O}_{w^{(1)}}(x_1) U(x_1, x_0) \mathcal{O}_{w^{(0)}}(x_0) U(x_0, x_{m-1}) \right) = \\
&= \int_{S_n \times C_{m-3}(x_0, x_1, x_2)} \int_{\mathcal{M}_\beta} \left[\wedge_{j=n}^1 V_2^{(j)} \right] \wedge \operatorname{tr} \wedge_{\alpha=m-1}^3 \left[W_1^{(\alpha)} \mathcal{U}(x_\alpha, x_{\alpha-1}) \right] \wedge \\
&W_0^{(2)}(x_2) \mathcal{U}(x_2, x_1) W_0^{(1)}(x_1) \mathcal{U}(x_1, x_0) W_0^{(0)}(x_0) \mathcal{U}(x_0, x_{m-1}) ,
\end{aligned} \tag{64}$$

where the path integral is performed over the instanton sector β . In the first expression, the integral over boundary insertions is performed over the set $C_{m-1}(x_0)$ of configurations $(x_1 \dots x_{m-1})$ such that $x_0, x_1, x_2 \dots x_{m-1}$ are distinct and cyclically ordered on ∂D , and over the set $S_{n-1}(z_0)$ of bulk insertions points $z_1 \dots z_n \in \operatorname{Int} D$ such that the points $z_0 \dots z_n$ are all distinct. A similar convention is used in the second formula.

We now proceed to give a proof of these equations. Instead of adapting the super-space arguments of [27], we give a direct analysis in the universal instanton formalism. This has the advantage that it does not assume that the string background satisfies the string equations of motion.

Recovering the nonlinear sigma model formulation requires that we fix the $SL(2, \mathbb{R})$ gauge in the string amplitude J_β . Since (60) gives the latter in terms of the moduli space $\mathcal{M}(\beta)$, it is not immediately clear what ‘fixing the gauge’ should mean. The solution is to express J_β in terms of forms defined on the moduli space $\tilde{\mathcal{M}}(\beta)$, on which the symmetry group $SL(2, \mathbb{R})$ acts nontrivially. For this, note that the target space forms $v^{(j)}$ and $w^{(\alpha)}$ can be pulled back to $\tilde{\mathcal{M}}(\beta)$ with the help of the evaluation maps. The resulting forms $\tilde{V}^{(j)}$ and $\tilde{W}^{(\alpha)}$ on $\tilde{\mathcal{M}}(\beta)$ are related to $\hat{V}^{(j)} := e_j^*(v^{(j)})$ and $\hat{W}^{(\alpha)} = e_\alpha^*(w^{(\alpha)})$ through:

$$\tilde{V}^{(j)} = p^*(\hat{V}^{(j)}) \tag{65}$$

$$\tilde{W}^{(\alpha)} = p^*(\hat{W}^{(\alpha)}) . \tag{66}$$

This follows from (58). We see that \tilde{V} and \tilde{W} are basic forms on the fibration $\tilde{\mathcal{M}}(\beta) \rightarrow \mathcal{M}(\beta)$ (i.e. pull-backs of forms defined over its base). Given a section s of this fibration, the property $p \circ s = \operatorname{id}_{\mathcal{M}(\beta)}$ implies that $s^*(\tilde{V}^{(j)}) = \hat{V}^{(j)}$ and $s^*(\tilde{W}^{(\alpha)}) = \hat{W}^{(\alpha)}$. Hence we can reformulate J_β as follows:

$$\begin{aligned}
J_\beta &= \int_{\tilde{\mathcal{M}}(\beta)} \left[\wedge_{j=n}^1 s^*(\tilde{V}^{(j)}) \right] \wedge \operatorname{tr} \left[\wedge_{\alpha=m-1}^0 \left(s^*(\tilde{W}^{(\alpha)}) \mathcal{U}_{\alpha, \alpha-1} \right) \right] = \\
&= \int_{\operatorname{Im} s} \left[\wedge_{j=n}^1 \tilde{V}^{(j)} \right] \wedge \operatorname{tr} \left[\wedge_{\alpha=m-1}^0 \left(\tilde{W}^{(\alpha)} \tilde{\mathcal{U}}_{\alpha, \alpha-1} \right) \right] .
\end{aligned} \tag{67}$$

To understand the precise relation between $\tilde{V}^{(j)}$, $\tilde{W}^{(\alpha)}$ and $V^{(j)}$, $W^{(\alpha)}$, consider the maps $p_j : \tilde{\mathcal{M}}(\beta) \rightarrow D \times \mathcal{M}_\beta$ and $p_\alpha : \tilde{\mathcal{M}}(\beta) \rightarrow \partial D \times \mathcal{M}_\beta$ defined through:

$$p_j(z, x, \phi) := (z_j, \phi) \quad (68)$$

$$p_\alpha(z, x, \phi) := (x_\alpha, \phi) \quad (69)$$

It is easy to see that one has:

$$\Phi \circ p_j = \tilde{e}_j (= e_j \circ p) \quad (70)$$

$$\Phi_\partial \circ p_\alpha = \tilde{e}_\alpha (= e_\alpha \circ p) \quad (71)$$

It follows that:

$$\tilde{V}^{(j)} = p_j^*(V^{(j)}) = p^*(\hat{V}^{(j)}) \quad (72)$$

$$\tilde{W}^{(\alpha)} = p_\alpha^*(W^{(\alpha)}) = p^*(\hat{W}^{(\alpha)}) \quad (73)$$

The relation between the various maps is summarized in the commutative diagrams below.

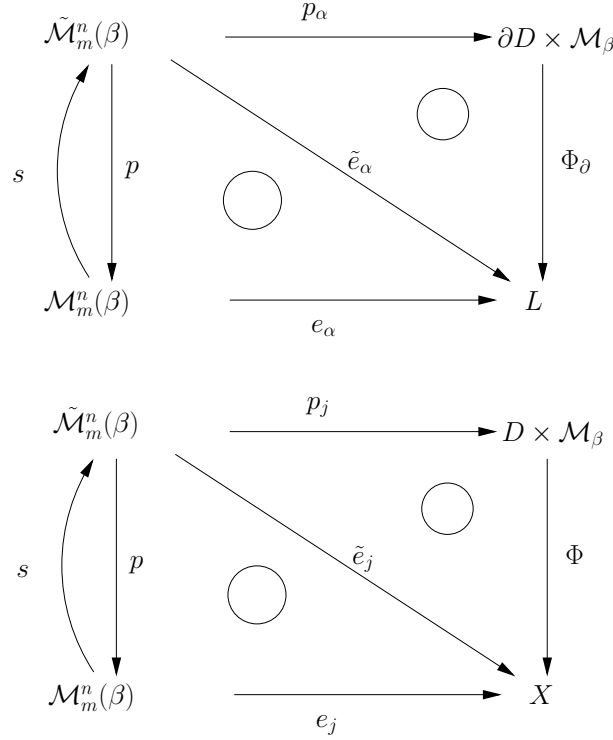


Figure 4. Projection and evaluation maps.

The section s implements a choice of gauge for the $SL(2, \mathbb{R})$ symmetry. To recover the nonlinear sigma model description, one simply picks a gauge in which the positions of some insertion points are fixed¹⁴. Let us show this explicitly for the case $m \geq 3$. In this situation, we can fix the positions of the first 3 boundary insertions to some values x_0, x_1, x_2 by choosing the following section of $\tilde{\mathcal{M}}(\beta) \rightarrow \mathcal{M}(\beta)$:

$$s_{x_0, x_1, x_2}(q) = (z_1^q(z_0, x_1, x_2) \dots z_n^q(x_0, x_1, x_2), x_0, x_1, x_2, x_3^q(x_0, x_1, x_2) \dots x_{m-1}^q(x_0, x_1, x_2), \phi_{x_0, x_1, x_2}^q) . \quad (74)$$

Here q is a point in $\mathcal{M}(\beta)$, i.e. an orbit of the $SL(2, \mathbb{R})$ action on $\tilde{\mathcal{M}}(\beta)$. In the right hand side, we have picked the unique point (z, x, ϕ) on this orbit with the property the the first 3 components of x coincide with x_0, x_1 and x_2 . This determines the remaining components $x_3 \dots x_{m-1}$, as well as $z_1 \dots z_n$ in terms of the orbit q and the fixed points x_0, x_1 and x_2 . It also determines a special instanton ϕ_{x_0, x_1, x_2}^q . Due to the form of the action (54), ϕ_{x_0, x_1, x_2}^q will cover the entire moduli space of instantons \mathcal{M}_β precisely once, as we let the orbit q cover $\mathcal{M}(\beta)$. Notice that the image of s_{x_0, x_1, x_2} coincides with the domain of integration of (64):

$$Im s_{x_0, x_1, x_2} = S_n \times C_{m-3}(x_0, x_1, x_2) \times \mathcal{M}_\beta . \quad (75)$$

Combining this observation with properties (72) and (67), it immediately follows that (60) implies (64). A similar argument applies for (63).

4.5 Exceptional string amplitudes and instanton corrections to the localized BRST operator

The disk string amplitudes containing one or two boundary insertions and no bulk insertion or one bulk insertion and no boundary insertions are special from a few points of view. These string correlators cannot be expressed as sigma model amplitudes of descendants, due to the fact that fixing the positions of operator insertions does not suffice to fix the $SL(2, \mathbb{R})$ gauge. In particular, one cannot use the sigma model non-renormalization results in order to conclude that the *string* amplitudes $\langle\langle \mathcal{O}_v \rangle\rangle$ and $\langle\langle \mathcal{O}_w \rangle\rangle$, $\langle\langle \mathcal{O}_{w_1} \mathcal{O}_{w_2} \rangle\rangle$ do not receive instanton corrections. As we discuss in [3], the presence of such amplitudes is related to the fact that the pair (L, A) gives a correct description of the topological D-brane only in the large radius limit. Since we express string amplitudes in terms of the large radius data (L, E) , we are building a topological string theory around a large radius vacuum, which generally differs from the true vacuum. The resulting string amplitudes can be brought to the standard form upon performing a shift from the large radius vacuum to the true vacuum [3].

Let us discuss the three exceptional classes of string amplitudes. We start with the **two-point boundary amplitude**, which has the instanton expression:

$$\langle\langle \mathcal{O}_{w_1} \mathcal{O}_{w_2} \rangle\rangle = \sum_{\beta} \langle\langle \mathcal{O}_{w_1} \mathcal{O}_{w_2} \rangle\rangle_{\beta} e^{-S_{\beta}} . \quad (76)$$

¹⁴Complete gauge-fixing via this procedure is not possible for string correlators containing a low number of insertions. Such correlators are exceptional and are discussed in the next subsection.

The coefficients $\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle_\beta$ in this expansion are defined through integrals over $\mathcal{M}_2^0(\beta)$ as in (60). One can fix the position of the two boundary insertion points at x_1 and x_2 as before, but this does not suffice to define a section of the fibration $\tilde{\mathcal{M}}_2^0(\beta) \rightarrow \mathcal{M}_2^0(\beta)$. Instead, this procedure gives a presentation of $\mathcal{M}_2^0(\beta)$ as a quotient of \mathcal{M}_β through the subgroup $G \approx \mathbb{R}$ of $SL(2, \mathbb{R})$ which fixes the points x_1 and x_2 . Hence one can express $\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle_\beta$ as an integral over $\mathcal{M}_\beta/\mathbb{R}$. This differs from the the nonlinear sigma model amplitude $\langle\mathcal{O}_{w_1}(x_1)\mathcal{O}_{w_2}(x_2)\rangle_\beta$, which is expressed as an integral over \mathcal{M}_β . In particular, the latter depends on the insertion points x_1 and x_2 while the former is independent of this choice.

The trivial instanton term in the expansion (76) is exceptional, since the action of $SL(2, \mathbb{R})$ on the moduli space $\tilde{\mathcal{M}}_2^0(0) = C^2 \times L$ is not fixed-point free. To recover the correct value of $\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle^{(0)}$, one must perform a direct analysis of localization starting with the Polyakov path integral¹⁵. Alternatively, one can use the results of [8], which studied this localization (in the Hamiltonian approach) for the trivial instanton sector¹⁶. It follows from the results of [8] that the relevant two-point amplitude in this sector must be:

$$\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle^{(0)} = \frac{1}{r} \int_L \text{tr}(w_1 \wedge d_A w_2) \quad . \quad (77)$$

(this gives the quadratic term of the large radius string field theory action of [8]). Note that (77) has no direct relation to the trivial instanton sector contribution to the sigma-model boundary two-point amplitude. In particular, the latter is non-vanishing if $\text{rank} w_1 + \text{rank} w_2 = d$, while the selection rule for (77) is $\text{rank} w_1 + \text{rank} w_2 = d - 1$.

The higher instanton corrections to (76) can be encoded by writing:

$$\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle = \rho(w_1, Q_o w_2) = \frac{1}{r} \int_L \text{tr}[w_1 \wedge Q_o w_2] \quad , \quad (78)$$

where $Q_o = \sum_\beta Q_o(\beta)$, with $Q_o(0) = d$ and $Q_o(\beta) : \Omega^*(L, E) \rightarrow \Omega^*(L, E)$ some degree +1 linear maps determined by:

$$\langle\langle\mathcal{O}_{w_1}\mathcal{O}_{w_2}\rangle\rangle^{(\beta)} = \rho(w_1, Q_o(\beta) w_2) = \frac{1}{r} \int_L \text{tr}[w_1 \wedge Q_o(\beta) w_2] \quad . \quad (79)$$

The fact that the higher instanton contributions $Q_o(\beta)$ ($\beta \neq 0$) need not vanish was noticed for the first time in [6]. The operator Q_o can be viewed as an instanton-corrected expression for the BRST charge of the model. To prevent misunderstanding

¹⁵In this approach, the appearance of the differential d_A in (77) is related to the existence of fixed points for the $SL(2, \mathbb{R})$ action on $\tilde{\mathcal{M}}_2^0(0)$.

¹⁶Most of the analysis of [8] is restricted to the case when $X = T^*M$ for some manifold M . This assumption is made in [8] with the sole purpose of eliminating instanton corrections. Indeed, all contributions from nontrivial instanton sectors vanish for such manifolds, due to a vanishing theorem proved in [8]. However, the entire analysis of [8] remains correct in the trivial instanton sector ($\beta = 0$) or, equivalently, in the large radius limit for an arbitrary Calabi-Yau manifold X .

¹⁷, let me make a few observations on the physical meaning of this interpretation. In [8], it was argued that the BRST charge of the model can be represented (in the large radius limit) as the differential d_A . This follows from a localization argument in the *trivial* instanton sector $\beta = 0$, and cannot be extended to nontrivial sectors. The fact that Q_o receives worldsheet instanton corrections refers to the realization of the BRST symmetry on the space $\Omega^*(\text{End}(E))$, i.e. it is a statement about the nature of localization. As we shall see in [3], the presence of such corrections modifies the string field action. It is important, however, to realize that such effects modify the *classical* BV action of the string field theory, and do not represent a quantum (i.e. loop) effect in string theory per se (even though they do represent a quantum effect on the *worldsheet*). We do *not* claim that the BRST operator receives corrections (and, in fact, may become ‘anomalous’) due to the quantum dynamics governed by the string field action¹⁸. Rather, we are saying that the *tree level* string field action of [8] is incomplete due to worldsheet effects which modify the localization formula for the BRST charge.

In fact, the localized BRST operator is given by a ‘shifted’ form Q'_o of Q_o . This further modification of Q_o is induced by the necessity of performing a shift of the string background, as I discuss in more detail in [3]. Indeed, the operator Q_o does not generally square to zero, as already noticed for the singly-wrapped case by Fukaya and collaborators [6, 7]. It is the modification $Q_o \rightarrow Q'_o$ which assures that the shifted operator Q'_o is nilpotent.

Finally, we consider the **one-point string amplitudes** $\langle\langle \mathcal{O}_v \rangle\rangle$ and $\langle\langle \mathcal{O}_w \rangle\rangle$. These can be written:

$$\langle\langle \mathcal{O}_v \rangle\rangle = (v, p_0^0) , \quad (80)$$

$$\langle\langle \mathcal{O}_w \rangle\rangle = \rho(w, q_0^0) , \quad (81)$$

for some elements $p_0^0 \in \Omega^3(X)$ and $q_0^0 \in \Omega^2(L, \text{End}(E))$. Here $(,)$ is a degree $2d - 1$ bilinear form which is induced from the bulk topological metric when restricting to the semirelative complex of the closed string state space [3]. Since this requires a slightly lengthy analysis (to be given in [3]), I will concentrate on the amplitude $\langle\langle \mathcal{O}_w \rangle\rangle$.

As before, one has instanton expansions:

$$\langle\langle \mathcal{O}_w \rangle\rangle = \sum_{\beta} \langle\langle \mathcal{O}_w \rangle\rangle_{\beta} e^{-S_{\beta}} \quad (82)$$

$$q_0^0 = \sum_{\beta} q_0^0(\beta) e^{-S_{\beta}} , \quad (83)$$

with $q_0^0(\beta) \in \Omega^2(L, \text{End}(E))$ defined through:

$$\langle\langle \mathcal{O}_w \rangle\rangle_{\beta} = \frac{1}{r} \int_{\mathcal{M}_1^0(\beta)} \text{tr}(e_0^*(w) \mathcal{U}_{0,0}) = \rho(w, q_0^0(\beta)) , \quad (84)$$

¹⁷I thank R. Roiban for pointing out this possible source of confusion.

¹⁸Such an effect may also occur, but I am not able to test that at this point.

where $\mathcal{U}_{0,0} \in \text{End}(\mathcal{E}_0)$ is induced by the holonomy operator $\tilde{\mathcal{U}}_{0,0} \in \text{End}(\tilde{\mathcal{E}}_0)$ associated with a curve which winds once around the cylinder $\partial D \times \mathcal{M}_\beta$. Upon fixing the boundary insertion point to x_o , the space $\mathcal{M}_1^0(\beta)$ can be presented as the quotient of \mathcal{M}_β through the two-dimensional stabilizer G of x_o in $SL(2, \mathbb{R})$. In the conformal case, this gives a moduli space of dimension $\dim \mathcal{M}_2^0(\beta) = d - 2 = d + 1 - 3$, which implies that $q_0^0(\beta)$ is a form of rank 2. For $\beta = 0$, the moduli space $\mathcal{M}_1^0(0)$ coincides with $\mathcal{M}_0 = L$, and no rank two element of $\Omega^*(L, \text{End}(E))$ can be produced in this manner¹⁹ Therefore, one must take $q_0^0(0) = 0$. In particular, the sums in (82) can be taken to run over nontrivial homotopy classes $\beta \neq 0$. It follows that the large radius limit of q_0^0 vanishes, and the presence of a nonzero q_0^0 is purely an instanton effect.

The fact that q_0^0 need not vanish was noticed for the first time in [6]. As I discuss in more detail in [3], the presence of this string product is a signal that the string background receives quantum corrections; according to (a slight generalization of) [17], this means that we are building a string field theory around the wrong vacuum. Note that the product q_0^0 need not be zero even if the cycle L is special Lagrangian. In the special Lagrangian case, this is responsible for the obstructions to D-brane deformations observed in [11, 12]. This zeroth order product can be eliminated by shifting the string background [6, 7, 3]. As for p_0^0 , it can also be eliminated by shifting the string background [3].

4.6 Homological formulation for the boundary sector of a singly-wrapped D-brane

When E is a line bundle on L , the boundary nonlinear sigma model and string amplitudes admit a homological formulation similar to that familiar from the closed string case. In this situation, the bundle $\text{End}(E) \approx E^* \otimes E$ is trivial, and hence $\text{End}(E)$ -valued forms on L become usual complex-valued forms. The connection induced by A on $\text{End}(E)$ is a flat connection on \mathcal{O}_L . While $\text{End}(E)$ is trivial, the parallel transport of A must still be taken into account.

The homological form of the boundary amplitudes results by considering forms w_α with delta-function support on submanifolds K_α of L , having codimension l_α (that is, w_α are currents associated to such submanifolds). Then the pull-backs of w_α by the various evaluation maps used in the universal instanton formalism have delta-function support on codimension l_α cycles $e_{x_\alpha}^*(K_\alpha)$ of \mathcal{M}_β and $e_\alpha^*(K_\alpha)$ of $\mathcal{M}_m^0(\beta)$. Integration of the wedge product over the moduli space amounts to intersection of the pulled-back cycles. The product is zero unless the common intersection of the pulled-back cycles is a discrete set of points. It follows that the localization formulae reduce to finite sums over these intersection points. At each intersection point ϕ , the parallel transport operators \mathcal{U} can be composed, giving the holonomy in the fiber of the pulled-back line

¹⁹Inserting the differential d_A is not allowed, since this would lead to a form of degree 3. That $q_0^0(0)$ must vanish also follows from the general structure of open string field theory [17] and the fact that our background does satisfy the string equations of motion at large radius (see below).

bundle \mathcal{E} above that point, which coincides with the holonomy $W_\beta = Pe^{-\int_{\partial D} \phi^*(A)}$ of $-\phi^*(A)$ around ∂D . This quantity depends only on the relative homotopy class β , since the closed curves $\phi(\partial D)$ and $\phi'(\partial D)$ are homotopic to each other in L if ϕ and ϕ' belong to β .

The net effect is to produce the intersection form of the pulled-back cycles, modulo the holonomy factor W_β . For the boundary sigma model amplitude, the end result is:

$$I_\beta(K_{m-1}..K_0) = \#(\cap_{\alpha=m-1}^0 (e_{x_\alpha})^*(K_\alpha))_{\mathcal{M}_\beta} W_\beta \quad , \quad (85)$$

while for the boundary string amplitude one obtains:

$$J_\beta(K_{m-1}..K_0) = \#(\cap_{\alpha=m-1}^0 e_\alpha^*(K_\alpha))_{\mathcal{M}_m^0(\beta)} W_\beta \quad . \quad (86)$$

This allows us to extract a homological formulation of string products. Indeed, the boundary topological metric corresponds to the intersection form on L :

$$\check{\rho}(K_\alpha, K_\beta) = \#(K_\alpha \cap K_\beta)_L = \int_L v^{(\alpha)} \wedge v^{(\beta)} \quad . \quad (87)$$

Hence we can write, for example:

$$J_\beta(K_m..K_0) = \check{\rho}(K_m, \check{r}_m(\beta)(K_{m-1}..K_0)) W_\beta \quad , \quad (88)$$

where $\check{r}_m(\beta)(K_{m-1}..K_0) = (e_m)_*(e_{m-1}^*(K_{m-1}) \cap \dots \cap e_0^*(K_0))$. This gives the dual formulation of the boundary string products:

$$\check{r}_m = \sum_{\beta} \check{r}_m(\beta) W_\beta e^{-S_\beta} \quad . \quad (89)$$

\check{r}_m is a linear map from $C_*(L)^{\otimes m}$ to $C_*(L)$. Relation (89) recovers the products introduced by Fukaya and collaborators [6, 7].

Let me end with a few technical remarks for the mathematically savvy reader. The ‘off-shell Poincare duality’ argument employed above is quite non-rigorous, due to technical difficulties involved in defining the intersection theory of chains. In particular, it is not obvious how the argument works for the case when two of the chains K_α coincide. This is related to a host of technical issues discussed in detail in [7], to which I refer the interested reader. Unfortunately, similar problems are encountered in the cohomological formulation (for the singly or multiply-wrapped case), when one attempts to define string products off-shell via dualization with respect to the topological metrics. It remains a difficult problem to provide a rigorous analysis in this framework.

Finally, let me make a few remarks on the possibility of a homological approach for the multiply-wrapped case. It is intuitively clear that recovering a homological approach requires the use of ‘bundle-valued currents’. This is to say that the homological formulation of the boundary state space must include the Chan-Paton degrees of freedom described by the local system (flat bundle) E . One could try to achieve this by considering chains K in L endowed with extra data represented by a section of $\text{End}(E)$ above K , but it is not clear to what extent such a description is equivalent to the cohomological approach proposed in this paper.

A Localized form of the BRST operator in the large radius limit

This appendix re-derives the BRST invariance condition for the boundary observables in the ‘Lagrangian’ framework of [9, 10]. The result is well-known from [8], but there seems to be some confusion on this point in the literature, so I include a detailed derivation for completeness.

To identify the boundary topological observables, it is most convenient to follow the ‘Lagrangian’ approach of [9, 10]. Let Q_o denote the boundary BRST generator, which satisfies $[Q_o, \phi] = -\chi$ and $\{Q_o, \chi\} = 0$. We want to express the constraint $\{Q_o, \mathcal{O}_w\} = 0$ as a condition on the bundle-valued form w . To understand the structure of (17), let us first look at its form in the vicinity of a given point $x \in \partial D$. Choosing suitable open neighborhoods U of $\phi(x)$ and $V \subset \phi^{-1}(U)$ of x , we can trivialize $E|_U$ with the help of a set $\{s_\alpha\}_{\alpha=1..r}$ of linearly independent sections²⁰. We also consider the dual frame $\{s_\alpha^*\}_{\alpha=1..r}$ of $E^*|_U$, which satisfies $s_\alpha^*(s_\beta) = \delta_{\alpha,\beta}$. Finally, we consider a coordinate chart of L on U , which defines tangent vectors $\{\partial_i\}_{i=1..d}$ to U . Let ∇ be the D-brane connection on E and A its matrix-valued coefficient one-form in the frame s :

$$d\nabla(s_\alpha) = A_{\beta\alpha}s_\beta \quad . \quad (90)$$

We will also need the dual connection ∇^* on E^* , which is defined through the condition:

$$X[\eta(s)] = [\nabla_X^*(\eta)](s) + \eta(\nabla_X(s)) \quad , \quad (91)$$

for any local sections η of E^* and s of E , and any vector field X on U . Its one-form coefficient matrix in the frame s^* is given by:

$$d\nabla^*(s_\alpha^*) = A_{\beta\alpha}^*s_\beta^* = -A_{\alpha\beta}s_\beta^* \quad , \quad (92)$$

i.e. $A^* = -A^t$.

With these preparations, we can write the bundle-valued form w as:

$$w = w^{\beta\alpha}s_\beta^* \otimes s_\alpha \quad , \quad (93)$$

and the observable \mathcal{O}_w :

$$\mathcal{O}_w(x) = w_{i_1..i_k}^{\beta\alpha}(\phi(x))s_\beta^*(\phi(x)) \otimes s_\alpha(\phi(x))\chi^{i_1}(x)...\chi^{i_k}(x) \quad , \quad (94)$$

where $w_{i_1..i_k}^{\beta\alpha} = w^{\beta\alpha}(\partial_{i_1}, \dots, \partial_{i_k})$.

Before attempting to compute the BRST variation of the object (17), we have to understand if we have actually provided a complete definition of what its BRST variation should be (as we shall see in a moment, the answer is negative). A complete definition of the model requires that we define how the operation Q acts on all ‘allowed’

²⁰Linearly independent at any point of U , i.e. a local frame for E above U .

configurations. The later are functionals of the basic worldsheet fields ϕ, χ, λ . The standard approach, followed for example in [9, 10, 8], is to specify the Q-variation of these basic fields and to define the action of Q on an arbitrary functional $F[\phi, \chi, \lambda]$ through functional differentiation:

$$(\delta_Q F)[\phi, \chi, \lambda] = \int_{\partial D} \left(\frac{\delta F}{\delta \phi(x)} (\delta_Q \phi)(x) + \frac{\delta F}{\delta \chi(x)} (\delta_Q \chi)(x) + \frac{\delta F}{\delta \lambda(x)} (\delta_Q \lambda)(x) \right) . \quad (95)$$

However, this assumes that are given a proper definition of the objects $\frac{\delta F}{\delta \phi}(x)$. Such a definition is not immediately obvious, since we want to functionally differentiate a geometrically nontrivial object F . Indeed, in our case $F[\phi, \chi] = \mathcal{O}_w$ is a bundle-valued form on ∂D . Its formal Q-variation is given by:

$$\delta_Q \mathcal{O}_w = i\xi \int_{\partial D} \frac{\delta \mathcal{O}_w}{\delta \phi(x)} \chi(x) . \quad (96)$$

To define the functional variation in the integrand, one must specify how to compare the values of the functional $f[\phi] = s_\beta^* \circ \phi \otimes s_\alpha \circ \phi$ for two different choices ϕ_1 and ϕ_2 of ϕ . In particular, we must know how to compare the values of $s_\beta^* \otimes s_\alpha$ at two *different* points $\phi_1(x)$ and $\phi_2(x)$ on L . The standard procedure for doing this is to consider a connection on the bundle $End(E)$ and compare the two values through the parallel transport it defines. Hence a complete *definition* of the action of Q in the boundary sector requires the use of a connection. There is only one natural candidate arising from our model's data, namely the connection induced by ∇ on $End(E)$. Hence we *define*:

$$\delta_Q s_\alpha = i\xi (d_\nabla s_\alpha)_x(\chi(x)) = i\xi \nabla_i s_\alpha(x) \chi^i(x) \quad (97)$$

$$\delta_Q s_\alpha^* = i\xi (d_{\nabla^*} s_\alpha^*)_x(\chi(x)) = i\xi \nabla_i^* s_\alpha^*(x) \chi^i(x) . \quad (98)$$

This allows us to compute the variation of $s_\alpha^* \circ \phi \otimes s_\beta \circ \phi$:

$$\delta_Q (s_\alpha^* \circ \phi \otimes s_\beta \circ \phi) = i\xi [(\nabla_i s_\alpha^*)(\phi(x)) \otimes s_\beta(\phi(x)) + s_\alpha(\phi(x)) \otimes (\nabla_i s_\beta)(\phi(x))] \chi^i(x) \quad (99)$$

$$= i\xi [-A_{\alpha\alpha'}{}^i(\phi(x)) \delta_{\beta'\beta} + \delta_{\alpha,\alpha'} A_{\beta'\beta,i}(\phi(x))] \chi^i(x) s_\beta^*(\phi(x)) \otimes s_\alpha(\phi(x)) . \quad (100)$$

One also has:

$$\delta_Q (w_{i_1 \dots i_k}^{\beta\alpha} \circ \phi)(x) = i\xi (\partial_i w_{i_1 \dots i_k}^{\beta\alpha})(\phi(x)) \chi^i(x) . \quad (101)$$

Combining these two results we obtain:

$$\{Q_o, \mathcal{O}_w\} = -\mathcal{O}_{d_A w} , \quad (102)$$

where d_A is the covariant differential induced by A on $End(E) \approx E^* \otimes E$:

$$d_A w = dw + [A, w] . \quad (103)$$

B The Maslow index

In this appendix I discuss the quantity $\mu(\beta)$ which appears in the virtual dimension of disk instanton moduli spaces [6]. Let us start by fixing a Lagrangian cycle L in X . Consider a map $\phi : D \rightarrow X$ from the unit disk to X , such that $s := \phi(\partial D) \subset L$. We let β denote its relative homotopy class in X with respect to L . Since D is contractible, one can globally trivialize the pulled back tangent bundle $\phi^*(TX)$ on D . This result also holds symplectically, i.e. the trivialization can be chosen such that it takes the symplectic form $\omega_x(x \in D)$ on each fiber into the standard symplectic form ω_d on $\mathbb{C}^d = \mathbb{R}^{2d}$. Here ω is the pulled-back Kahler form of X . Since L is Lagrangian, the tangent spaces $\phi^*(TL)_x$ for $x \in \partial D$ are Lagrangian subspaces of $(\phi^*(TX)_x, \omega_x)$. The symplectic trivialization of $\phi^*(TX)$ allows us to view the collection of such subspaces as a curve Γ in the Lagrangian Grassmannian $Lagr_d$. The latter is defined as the set of all Lagrangian subspaces of the symplectic vector space $(\mathbb{R}^{2d}, \omega_d)$ ²¹. This can be formalized by considering the *Gauss map* $G : L \rightarrow Lagr(TX)$, which associates to each point u of L the tangent bundle $T_u L \subset T_u X$. By using the trivialization of $\phi^*(X)$, this induces a map $g : \partial D \rightarrow Lagr_d$. The curve Γ is the image of ∂D through g .

It is a classical result that $\pi_1(Lagr_d) \approx \mathbb{Z}$, with a generator which we shall call γ . Hence the homotopy class of the curve Γ inside of $Lagr_d$ can be expressed as a multiple of this generator:

$$\Gamma = \mu(\phi)\gamma \quad . \quad (104)$$

The integer μ is the *Maslow index* of ϕ . This number is independent of the trivialization chosen for $\phi^*(TX)$, since a change of this trivialization induces a homotopy transformation of Γ . Moreover, the Maslow index of ϕ depends only on the relative homotopy class β , since two maps ϕ_1, ϕ_2 in the same relative homotopy class induce homotopic curves Γ_1, Γ_2 in $Lagr_d$. Hence we can write $\mu(\phi) = \mu(\beta)$.

Intuitively, the index tells us how many times the tangent space $T_u L$ ‘rotates’ inside of $T_u X$ as u moves once around the closed curve $s = \phi(\partial D)$. This quantity appears in the index theorem of [6, 7] because the boundary conditions for χ along ∂D require it to be a section of $\phi^*(TL)$. Hence the ‘winding’ of $T_u L$ ($u \in \phi(\partial D)$) inside of $T_u X$ is related to the ‘winding’ of the fermionic section $\chi|_{\partial D}$ as one follows the boundary of the disk.

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²¹A Lagrangian subspace V of a symplectic vector space is a subspace whose dimension is half that of the ambient space and such the the restriction of the symplectic form to that subspace is zero.

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